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Lepton mixing matrix in standard model extended by one sterile neutrino

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Abstract. We consider the simplest extension of the standard electroweak model by one sterile neutrino that allows for neutrino masses and mixing. We find that its leptonic sector contains much less free physical parameters than previously realized. In addition to the two neutrino masses, the lepton mixing matrix in charged current interactions involves $(n-1)$ free physical mixing angles for n generations. The mixing matrix in neutral current interactions of neutrinos is completely fixed by the two masses. Both interactions conserve CP. We illustrate the phenomenological implications of the model by vacuum neutrino oscillations, tritium β decay and neutrinoless double β decay. It turns out that, due to the revealed specific structure in its mixing matrix, the model with any n generations cannot accommodate simultaneously the data by KamLAND, K2K and CHOOZ.

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1 Introduction

The experiments with solar, atmospheric, reactor and accelerator neutrinos have now provided evidence that neutrinos have mass and mix; for recent reviews on neutrino oscillations, see e.g. [1, 2], and for an introduction to neutrino physics, see e.g. [3]. These experimental results can be best understood in terms of neutrino oscillations [4–7]. This implies unambiguously that the standard model (SM) of the electroweak interactions has to be extended at least in the leptonic sector. Most phenomenological analyses performed so far assume three active massive neutrinos and a three by three unitary mixing matrix for charged current (CC) interactions of leptons. The leptonic sector then contains two differences of neutrino masses squared, three mixing angles plus three CP-violating phases. It has been shown to be capable of accommodating all neutrino data except the LSND result [8–10], which however was not confirmed by other short baseline experiments [11–13] and remains to be clarified by MiniBooNE [14] in the future. From the model building point of view, such a scenario requires new degrees of freedom to be added to SM, either some heavy neutrinos [15–17], or a Higgs triplet which also develops a vacuum expectation value [18]. In the first case, the three by three mixing matrix amongst three light, active neutrinos is only approximately unitary to the extent that their mixing with heavy neutrinos can be ignored in the analysis of the current data. Thus it amounts to a low energy effective theory of a fundamental one which could be much more complicated. In the second one, the existence of a non-doublet Higgs boson is always severely constrained by precision electroweak data and the null result of direct Higgs searches.

There are also attempts to incorporating the LSND result by including explicitly a sterile neutrino into the mixing scheme which was introduced earlier in the other context [19–23]. But they are found to be disfavored by the experimental data either because of the tension between the positive result of LSND and the negative ones by other short baseline experiments or because of the rejection of sterile neutrinos' involvement in solar and atmospheric data [24].

In this work, we put aside the LSND result as in most studies and ask whether it is possible to understand the neutrino data in a minimal extension of SM. The extension is minimal in the sense that it introduces the least numbers of new degrees of freedom and free physical parameters into the fundamental theory without endangering precision electroweak data. With this in mind, one possibility would be to introduce one sterile neutrino to the SM of three generations. Such a kind of models were systematically studied a long time ago in the pioneering work of [25]. They were also considered without including Majorana mass terms in [26, 27]. According to the analysis in [25], this would introduce five mixing angles and three CP-violating phases, in addition to two neutrino masses (with the other two being massless). There should thus be some room to accommodate the mentioned neutrino data that essentially call for two mass-squared differences and three independent mixing angles. However, as we shall ana-

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lyse in the next section, there are actually only two mixing angles and there is no CP-violating phase in the lepton mixing matrix after we take into account completely the constraints from the texture zero that appears in the original neutrino mass matrix [28, 29]. Furthermore, as we shall show in Sect. 3, due to the special structure in the mixing matrix exposed in Sect. 2, it is even not possible to accommodate simultaneously the data by KamLAND, K2K and CHOOZ [30–34] for any number of n generations, although in that case there are $(n-1)$ free mixing angles at our disposal. The same structure also results in the vanishing leading contribution to neutrinoless double β decay for any n which in principle is allowed to occur due to the Majorana nature of the neutrinos. Our results are summarized and conclusions are drawn in the last section.

2 Parametrization of mixing matrix

In this section we first describe the leptonic sector of the n-generation SM extended by n_0 sterile neutrinos. Then, we specialize to the simplest case of $n_0=1$ to parametrize the lepton mixing matrix and count its independent physical parameters.

2.1 The model

The only new fields compared to SM are the n_0 sterile neutrinos that we choose to be right-handed without loss of generality, $s_{\mathrm{R}x}$, $x=1,\cdots,n_0$. It is sufficient for us to concentrate on the leptonic sector of the model that contains the standard n generations of the doublets, $F_{\mathrm{L}a}=(n_{\mathrm{L}a},f_{\mathrm{L}a})^{\mathrm{T}}$, and of the charged lepton singlets, $f_{\mathrm{R}a}$, $a=1,\cdots,n$. Here L, R refer to the left- and right-handed projections of the fields in terms of $P_{\mathrm{L,R}}=(1\mp\gamma^5)/2$. The kinetic and gauge interaction terms of the leptons are

$$\begin{aligned}
\mathcal{L}_k &= \overline{n_{\mathrm{L}a}}\mathrm{i}\partial\!\!\!/ n_{\mathrm{L}a}+\overline{s_{\mathrm{R}x}}\mathrm{i}\partial\!\!\!/ s_{\mathrm{R}x}+\overline{f_a}\mathrm{i}\partial\!\!\!/ f_a\\
\mathcal{L}_{\mathrm{CC}} &= \frac{g}{\sqrt{2}}\left[W^+_\mu\overline{n_{\mathrm{L}a}}\gamma^\mu f_{\mathrm{L}a}+W^-_\mu\overline{f_{\mathrm{L}a}}\gamma^\mu n_{\mathrm{L}a}\right]\\
\mathcal{L}_{\mathrm{NC}} &= \frac{g}{2c_{\mathrm{W}}}Z_\mu\left[\overline{n_{\mathrm{L}a}}\gamma^\mu n_{\mathrm{L}a}+\overline{f_a}\gamma^\mu\left(-P_{\mathrm{L}}+2s^2_{\mathrm{W}}\right)f_a\right]\\
\mathcal{L}_{\mathrm{EM}} &= -eA_\mu\overline{f_a}\gamma^\mu f_a\,,
\end{aligned}\tag{1}$$

where summation over the indices a,x is implied. g,e are respectively the couplings of $SU(2)_{\mathrm{L}}$ and $U(1)_{\mathrm{EM}}$, and $c_{\mathrm{W}}=m_W/m_Z$ with $m_{W,Z}$ being the masses of $W^\pm,Z$.

Since the sterile neutrinos are neutral under $SU(2)_{\mathrm{L}}\times U(1)_{\mathrm{Y}}$ by definition, they can have bare mass terms of Majorana type,

$$-\mathcal{L}_{s_{\mathrm{R}}}=\frac{1}{2}M_{xy}\overline{s^C_{\mathrm{R}x}}s_{\mathrm{R}y}+\frac{1}{2}M^*_{xy}\overline{s_{\mathrm{R}y}}s^{\mathrm{C}}_{\mathrm{R}x}\tag{2}$$

where $\psi^{\mathrm{C}}=\mathcal{C}\gamma^0\psi^*$ stands for the charge-conjugate field of ψ with $\mathcal{C}=\mathrm{i}\gamma^0\gamma^2$ satisfying $\mathcal{C}=-\mathcal{C}^\dagger=-\mathcal{C}^{\mathrm{T}}=-\mathcal{C}^{-1}$ and $\mathcal{C}\gamma^{\mu\mathrm{T}}\mathcal{C}=\gamma^\mu$. We denote $s^{\mathrm{C}}_{\mathrm{R}}=(s_{\mathrm{R}})^{\mathrm{C}}$ for brevity. The $n_0\times n_0$ complex matrix M is symmetric due to anticommutativity of fermion fields, but it is otherwise general. Together with the Yukawa terms of the leptons,

$$-\mathcal{L}_{\mathrm{Y}}=y^f_{ab}\overline{F_{\mathrm{L}a}}\varphi f_{\mathrm{R}b}+y^n_{ax}\overline{F_{\mathrm{L}a}}\tilde{\varphi}s_{\mathrm{R}x}+\mathrm{h.c.}\tag{3}$$

where φ is the Higgs doublet field that develops a vacuum expectation value, $\langle\varphi\rangle=(0,1)^{\mathrm{T}}v/\sqrt{2}$, and $\tilde{\varphi}=\mathrm{i}\sigma^2\varphi^*$, the lepton mass terms become

$$-\mathcal{L}_m=\left[\overline{f_{\mathrm{L}}}m^f f_{\mathrm{R}}+\overline{n_{\mathrm{L}}}Ds_{\mathrm{R}}+\mathrm{h.c.}\right]+\frac{1}{2}\left[\overline{s^{\mathrm{C}}_{\mathrm{R}}}Ms_{\mathrm{R}}+\mathrm{h.c.}\right],\tag{4}$$

where $m^f=y^fv/\sqrt{2}$ and $D=y^nv/\sqrt{2}$ are $n\times n$ and $n\times n_0$ complex matrices respectively. The charged lepton mass terms are diagonalized as usual by biunitary transformations,

$$f_{\mathrm{L}}=X_{\mathrm{L}}\ell_{\mathrm{L}},\quad f_{\mathrm{R}}=X_{\mathrm{R}}\ell_{\mathrm{R}},\quad X^{-1}_{\mathrm{L,R}}=X^\dagger_{\mathrm{L,R}}\,,\tag{5}$$

with

$$X^\dagger_{\mathrm{L}}m^fX_{\mathrm{R}}=m^\ell=\mathrm{diag}(m_e,m_\mu,m_\tau,\cdots)\,,\tag{6}$$

being real and positive. We shall denote the mass eigenstate fields of the charged leptons, $\ell_{\mathrm{L,R}}$, by Greek indices, $\alpha,\beta=1,2,\cdots,n$, corresponding to the electron, muon, etc.

The kinetic and mass terms for the neutral leptons contain the fields n_{L}, s_{R} and $s^{\mathrm{C}}_{\mathrm{R}}$. To diagonalize, we first rewrite them uniformly in terms of the fields n_{L}, s_{R} and their charge-conjugates $n^{\mathrm{C}}_{\mathrm{L}}$, $s^{\mathrm{C}}_{\mathrm{R}}$. Using $\overline{\psi^{\mathrm{C}}}\mathrm{i}\gamma^\mu\partial_\mu\chi^{\mathrm{C}}=-\mathrm{i}\partial_\mu(\bar{\chi}\gamma^\mu\psi)+\bar{\chi}\mathrm{i}\gamma^\mu\partial_\mu\psi$, where the total derivative term can be ignored in the Lagrangian, and $\overline{\psi^{\mathrm{C}}}\chi^{\mathrm{C}}=\bar{\chi}\psi$, the terms become

$$\begin{aligned}
\mathcal{L}^\nu_k &= \frac{1}{2}\left(\overline{n^{\mathrm{C}}_{\mathrm{L}}},\overline{s_{\mathrm{R}}}\right)\mathrm{i}\partial\!\!\!/\begin{pmatrix}n^{\mathrm{C}}_{\mathrm{L}}\\ s_{\mathrm{R}}\end{pmatrix}+\frac{1}{2}\left(\overline{n_{\mathrm{L}}},\overline{s^{\mathrm{C}}_{\mathrm{R}}}\right)\mathrm{i}\partial\!\!\!/\begin{pmatrix}n_{\mathrm{L}}\\ s^{\mathrm{C}}_{\mathrm{R}}\end{pmatrix},\\
-\mathcal{L}^\nu_m &= \frac{1}{2}\left(\overline{n_{\mathrm{L}}},\overline{s^{\mathrm{C}}_{\mathrm{R}}}\right)m^n\begin{pmatrix}n^{\mathrm{C}}_{\mathrm{L}}\\ s_{\mathrm{R}}\end{pmatrix}+\frac{1}{2}\left(\overline{n^{\mathrm{C}}_{\mathrm{L}}},\overline{s_{\mathrm{R}}}\right)m^{n\dagger}\begin{pmatrix}n_{\mathrm{L}}\\ s^{\mathrm{C}}_{\mathrm{R}}\end{pmatrix},
\end{aligned}\tag{7}$$

where the $(n+n_0)$-dimensional, symmetric mass matrix in the new basis is

$$m^n=\begin{pmatrix}0_n & D\\ D^{\mathrm{T}} & M\end{pmatrix},\tag{8}$$

with 0_n being the zero matrix of n dimensions. For $n>n_0$, which covers our case of interest, $n=3$, $n_0=1$ later on, it contains a zero eigenvalue of degeneracy $(n-n_0)$ and $2n_0$ eigenvalues which are non-zero and nondegenerate for general parameters D, M. Without changing the diagonal form of the kinetic terms, we make a unitary transformation

$$\begin{pmatrix}n^{\mathrm{C}}_{\mathrm{L}}\\ s_{\mathrm{R}}\end{pmatrix}=Y\nu_{\mathrm{R}},\quad Y^{-1}=Y^\dagger,\tag{9}$$

which also fixes the transformation of the conjugate fields,

$$\begin{pmatrix}n_{\mathrm{L}}\\ s^{\mathrm{C}}_{\mathrm{R}}\end{pmatrix}=Y^*\nu^{\mathrm{C}}_{\mathrm{R}},\tag{10}$$

such that

$$Y^{\mathrm{T}} m^n Y = m^\nu = \mathrm{diag}(0,\cdots,0,m_{n-n_0+1},\cdots,m_{n+n_0})\,, \tag{11}$$

with the nonvanishing masses being real and positive. We shall denote the mass eigenstate fields of the neutral leptons, ν_{R}, by Latin indices, $j,k=1,2,\cdots,n+n_0$. Then

$$\begin{aligned}\mathcal{L}_k^\nu+\mathcal{L}_m^\nu &= \frac{1}{2}\left(\overline{\nu_{\mathrm{R}}}\mathrm{i}\partial\!\!\!/\nu_{\mathrm{R}}+\overline{\nu_{\mathrm{R}}^{\mathrm{C}}}\mathrm{i}\partial\!\!\!/\nu_{\mathrm{R}}^{\mathrm{C}}\right)\\ &\quad-\frac{1}{2}\left(\overline{\nu_{\mathrm{R}}^{\mathrm{C}}}m^\nu\nu_{\mathrm{R}}+\overline{\nu_{\mathrm{R}}}m^\nu\nu_{\mathrm{R}}^{\mathrm{C}}\right)\,,\end{aligned} \tag{12}$$

which may be put in the compact form

$$\mathcal{L}_k^\nu+\mathcal{L}_m^\nu=\frac{1}{2}\bar{\nu}\left(\mathrm{i}\partial\!\!\!/-m^\nu\right)\nu \tag{13}$$

by introducing the Majorana neutrino fields

$$\nu=\nu_{\mathrm{R}}+\nu_{\mathrm{R}}^{\mathrm{C}}\,, \tag{14}$$

satisfying $\nu^{\mathrm{C}}=\nu$.

Now we express the interactions of leptons in terms of the fields with a definite mass. There are no changes in $\mathcal{L}_{\mathrm{NC}}^\ell+\mathcal{L}_{\mathrm{EM}}^\ell$ for the charged leptons. The CC interaction becomes

$$\mathcal{L}_{\mathrm{CC}}=\frac{g}{\sqrt{2}}\left[V_{\beta j}^{\mathrm{C}}W_\mu^-\overline{\ell_{\mathrm{L}\beta}}\gamma^\mu\nu_j+V_{\beta j}^{\mathrm{C}*}W_\mu^+\overline{\nu_j}\gamma^\mu\ell_{\mathrm{L}\beta}\right]\,, \tag{15}$$

where the $n\times(n+n_0)$ matrix

$$V_{\beta j}^{\mathrm{C}}=\sum_{a=1}^{n}\left(X_{\mathrm{L}}^\dagger\right)_{\beta a}Y_{a\,j}^* \tag{16}$$

is the leptonic analog of the mixing matrix $V_{\mathrm{CKM}}^\dagger$ in the hadronic sector. It is important to notice that only the first n rows in Y are involved in the CC mixing matrix V^{C} [25], because the remaining n_0 rows are associated with the sterile neutrinos s_{R} which do not enter any interactions. Due to this and unitarity of X_{L} and Y, we have

$$V^{\mathrm{C}}V^{\mathrm{C}\dagger}=1_n, \tag{17}$$

but $V^{\mathrm{C}\dagger}V^{\mathrm{C}}\neq 1_{n+n_0}$. Actually, the latter appears in the neutral current (NC) interaction for the neutrinos

$$\mathcal{L}_{\mathrm{NC}}^\nu=\frac{g}{2c_{\mathrm{W}}}V_{kj}^{\mathrm{N}}Z_\mu\overline{\nu_k}\gamma^\mu P_{\mathrm{L}}\nu_j \tag{18}$$

by the relation [25]

$$\begin{aligned}V_{kj}^{\mathrm{N}}&=\sum_{a,b=1}^{n}Y_{bk}\delta_{ba}Y_{aj}^*=\sum_{a,b=1}^{n}\sum_{\alpha}Y_{bk}\left(X_{\mathrm{L}}\right)_{b\alpha}\left(X_{\mathrm{L}}^\dagger\right)_{\alpha a}Y_{aj}^*\\ &=\left(V^{\mathrm{C}\dagger}V^{\mathrm{C}}\right)_{kj}\,.\end{aligned} \tag{19}$$

Using $\overline{\psi^{\mathrm{C}}}\gamma^\mu P_{\mathrm{L}}\chi^{\mathrm{C}}=-\bar{\chi}\gamma^\mu P_{\mathrm{R}}\psi$, $\nu_j^{\mathrm{C}}=\nu_j$ and hermiticity of V^{N}, the interaction can also be cast in the form

$$\mathcal{L}_{\mathrm{NC}}^\nu=\frac{g}{4c_{\mathrm{W}}}Z_\mu\overline{\nu_k}\gamma^\mu\left[\mathrm{i}\mathrm{Im}V_{kj}^{\mathrm{N}}-\gamma^5\mathrm{Re}V_{kj}^{\mathrm{N}}\right]\nu_j \tag{20}$$

In addition to the condition (17), V^{C} satisfies a relation that will be important in its parametrization. The original neutrino mass matrix m^n has an n-dimensional zero submatrix in its left-upper corner which is protected by gauge symmetry of the model. Then (11) implies that

$$\left(Y^*m^\nu Y^\dagger\right)_{ab}=0,\quad a,b=1,\cdots,n \tag{21}$$

Multiplying it by $(X_{\mathrm{L}}^\dagger)_{\alpha a}(X_{\mathrm{L}}^\dagger)_{\beta b}$, summing over a, b and using (16) leads to the matrix relation [28, 29]

$$V^{\mathrm{C}}m^\nu V^{\mathrm{C}T}=0_n\,. \tag{22}$$

Note that the above holds irrespective of $n>n_0$ when there are massless modes or $n\le n_0$ when there is none for the general parameters D, M.

2.2 The mixing matrix

From now on, we shall restrict ourselves to the case of $n_0=1$. We offer two ways to construct the mixing matrix V^{C}, one by using the constraints (17), (22) and the other by explicit diagonalization. This is then followed by counting the independent physical parameters contained in it.

We start with the case of $n=1$ in which the two neutrinos are generally massive and nondegenerate with masses denoted by $0<m_-<m_+$. The mixing matrix is a one-row complex matrix, $V^{\mathrm{C}}=(r_1\mathrm{e}^{\mathrm{i}\rho_1},\ r_2\mathrm{e}^{\mathrm{i}\rho_2})$. Equations (17) and (22) read

$$r_1^2+r_2^2=1,\quad m_-r_1^2+m_+r_2^2\mathrm{e}^{2\mathrm{i}(\rho_2-\rho_1)}=0\,, \tag{23}$$

whose solution gives

$$\begin{aligned}&V^{\mathrm{C}}=\mathrm{e}^{\mathrm{i}\rho_1}(c_m,\pm\mathrm{i}s_m),\\ &c_m=\sqrt{\frac{m_+}{m_++m_-}},\quad s_m=\sqrt{\frac{m_-}{m_++m_-}}\,,\end{aligned} \tag{24}$$

and then

$$V^{\mathrm{N}}=\begin{pmatrix}c_m^2 & \pm\mathrm{i}c_ms_m\\ \mp\mathrm{i}c_ms_m & s_m^2\end{pmatrix}\,. \tag{25}$$

The global phase in V^{C} can be absorbed into the electron field, while the two signs can be interchanged by flipping the sign of the field ν_2. Both mixing matrices are thus uniquely fixed by the two neutrino masses, in contrast to the claim in [25] that there are one free mixing angle and one free CP-violating phase. Note that the nonreality of $V^{\mathrm{C,N}}$ is not a sign of CP violation. If we want, we can absorb the i into the field ν_2 without changing its mass term so that the $V^{\mathrm{C,N}}$ are real. But this will introduce a nontrivial "creation phase" into it which was unity before rephasing [35–37].

The above result may also be obtained by direct diagonalization of the mass matrix in (8) where D and M are two complex numbers. The unitary matrix Y in (11) may be parametrized generally as

$$Y=\mathrm{e}^{\mathrm{i}\gamma_0}\begin{pmatrix}c\mathrm{e}^{\mathrm{i}\gamma_1} & s\mathrm{e}^{\mathrm{i}\gamma_2}\\ -s\mathrm{e}^{-\mathrm{i}\gamma_2} & c\mathrm{e}^{-\mathrm{i}\gamma_1}\end{pmatrix} \tag{26}$$

with $c^2+s^2=1$ and all parameters being real. Denoting $D=|D|\mathrm{e}^{\mathrm{i}\delta_1}$, $M=|M|\mathrm{e}^{\mathrm{i}\delta_2}$, (11) then yields

$$m_\pm=\frac{1}{2}\left[\sqrt{|M|^2+4|D|^2}\pm|M|\right]$$
$$\mathrm{e}^{\mathrm{i}2\gamma_1}=-\mathrm{e}^{\mathrm{i}2\gamma_2}=\pm\mathrm{i}\mathrm{e}^{-\mathrm{i}(\delta_1-\delta_2)},\quad \mathrm{e}^{\mathrm{i}2\gamma_0}=\pm\mathrm{i}\mathrm{e}^{-\mathrm{i}\delta_1} \quad (27)$$

and c, s being identified with c_m, s_m in (24). Since only the first row of Y appears in V^{C}, the relevant phase is $\mathrm{e}^{\mathrm{i}(\gamma_2-\gamma_1)}=\pm\mathrm{i}$ independently of $\delta_{1,2}$, and the preceding result is reproduced.

For the case of $n>n_0=1$, we generally have two massive modes with masses $m_-<m_+$, that we arrange to be the last two in the order of increasing mass, and $(n-1)$ massless modes. Assuming $(V^{\mathrm{C}})_{\alpha,j}=r_{\alpha,j}\mathrm{e}^{\mathrm{i}\rho_{\alpha,j}}$, (22) implies that

$$m_-r_{\alpha,n}r_{\beta,n}\mathrm{e}^{\mathrm{i}(\rho_{\alpha,n}+\rho_{\beta,n})}+m_+r_{\alpha,n+1}r_{\beta,n+1}\mathrm{e}^{\mathrm{i}(\rho_{\alpha,n+1}+\rho_{\beta,n+1})}=0\,. \quad (28)$$

Note that the constraints from the off-diagonal elements $(\alpha\neq\beta)$ are not independent but just resolve the separate two-fold ambiguities in the constraints for the diagonal elements $(\alpha=\beta)$ to an overall two-fold ambiguity:

$$\frac{r_{\alpha,n+1}}{r_{\alpha,n}}=\sqrt{r_m},\quad \mathrm{e}^{\mathrm{i}\rho_{\alpha,n+1}}=\pm\mathrm{i}\mathrm{e}^{\mathrm{i}\rho_{\alpha,n}} \quad (29)$$

with $r_m=\frac{m_-}{m_+}$. V^{C} can then be factorized as

$$V^{\mathrm{C}}=VU\,, \quad (30)$$

where V is an $n\times n$ matrix whose αth row is

$$\left(r_{\alpha,1}\mathrm{e}^{\mathrm{i}\rho_{\alpha,1}},\cdots,r_{\alpha,n-1}\mathrm{e}^{\mathrm{i}\rho_{\alpha,n-1}},r_{\alpha,n}\sqrt{1+r_m}\mathrm{e}^{\mathrm{i}\rho_{\alpha,n}}\right) \quad (31)$$

and U is an $n\times(n+1)$ matrix

$$U=\begin{pmatrix}1_{n-1} & 0_{(n-1)\times1} & 0_{(n-1)\times1}\\ 0_{1\times(n-1)} & c_m & \pm\mathrm{i}s_m\end{pmatrix}, \quad (32)$$

with c_m, s_m again given in (24). $UU^\dagger=1_n$ together with (17) gives $VV^\dagger=1_n$. Since V is a square matrix whose entries are otherwise arbitrary, it is a general unitary matrix of dimension n. The above structure yields

$$V^{\mathrm{N}}=\begin{pmatrix}1_{n-1} & & \\ & c_m^2 & \pm\mathrm{i}c_ms_m\\ & \mp\mathrm{i}c_ms_m & s_m^2\end{pmatrix}. \quad (33)$$

Thus the NC interactions of neutrinos do not contain any free mixing angles or CP-violating phases that may appear in their CC interactions. They conserve CP, and the off-diagonal interactions occur only between the two massive neutrinos.

This result can be confirmed by direct diagonalization of the mass matrix in (8). In the first step, we decouple the $(n-1)$ massless modes. Now D is an n-row column matrix whose entries are parametrized as $D_a=|D_a|\mathrm{e}^{\mathrm{i}\delta_a}$. The magnitudes define a vector in $\mathcal{R}^n$, which is rotated to the nth axis by the rotation R_0. Denoting $E_0=\mathrm{diag}(\mathrm{e}^{-\mathrm{i}\delta_1},\cdots,\mathrm{e}^{-\mathrm{i}\delta_n})$ and the $(n+1)$-dimensional unitary matrix

$$Y_0=\begin{pmatrix}(R_0E_0)^{\mathrm{T}} & \\ & 1\end{pmatrix}, \quad (34)$$

we have

$$Y_0^{\mathrm{T}}m^nY_0=\begin{pmatrix}0_{n-1} & & \\ & 0 & |D|\\ & |D| & M\end{pmatrix} \quad (35)$$

with $|D|=\sqrt{\sum_a|D_a|^2}$. The problem has thus been reduced to the case of $n=n_0=1$ that we treated earlier. The required 2-dimensional unitary matrix, now denoted as y_1, for diagonalizing the submatrix is given in (26) and (27). Then

$$Y^{\mathrm{T}}m^nY=\mathrm{diag}(0,\cdots,0,m_-,m_+) \quad (36)$$

where $Y=Y_0Y_1$ and

$$Y_1=\begin{pmatrix}1_{n-1} & \\ & y_1\end{pmatrix}. \quad (37)$$

Noting the block diagonal form of $Y_{0,1}$, (16) gives the same factorized form of V^{C} as in (30) with the same U as in (32), but now

$$V=X_{\mathrm{L}}^\dagger(R_0E_0)^\dagger\mathrm{e}^{\mathrm{i}(\gamma_0+\gamma_1)}\,. \quad (38)$$

Since X_{L} is a general unitary matrix, so is V.

Now we count the free physical parameters contained in the matrix V^{C}. An n-dimensional unitary matrix V may be parametrized as a product in any arbitrarily specified order of the n phase matrices, $e_\alpha(u_\alpha)$ $(\alpha=1,\cdots,n)$, and the $n(n-1)/2$ complex rotation matrices in the (α,β) plane, $\omega_{\alpha\beta}(\theta_{\alpha\beta},\varphi_{\alpha\beta})$ $(n\geq\beta>\alpha\geq1)$ [25]. Here $e_\alpha(z)$ is obtained by replacing the αth entry in 1_n by the phase z, and

$$\omega_{\alpha\beta}(\theta_{\alpha\beta},\varphi_{\alpha\beta})=e_\alpha(\mathrm{e}^{\mathrm{i}\varphi_{\alpha\beta}})R_{\alpha\beta}(\theta_{\alpha\beta})e_\alpha(\mathrm{e}^{-\mathrm{i}\varphi_{\alpha\beta}}) \quad (39)$$

where $R_{\alpha\beta}(\theta_{\alpha\beta})$ is the usual real rotation matrix through angle $\theta_{\alpha\beta}$ in the (α,β) plane. We choose the order of products in such a way that it fits our purpose here:

$$V=e_n(u_n)\prod_{\alpha=1}^{n-1}\omega_{\alpha n}(\theta_{\alpha n},\varphi_{\alpha n})\begin{pmatrix}X & \\ & 1\end{pmatrix}\equiv V_0\begin{pmatrix}X & \\ & 1\end{pmatrix}, \quad (40)$$

where X is the general $(n-1)$-dimensional unitary matrix containing $(n-1)^2$ real parameters and the $\omega_{\alpha n}$ factors are ordered from left to right in increasing α. From the revealed structure in (30), the matrix containing X can be pushed through the matrix U to be absorbed into the massless neutrino fields without causing any other changes in the Lagrangian. This leaves us with $(2n-1)$ real parameters in V_0.

However, not all parameters in V_0 are physical. For brevity, we denote $\omega_{\alpha n}(\theta_{\alpha n},\varphi_{\alpha n})=\omega_\alpha$, $R_{\alpha n}(\theta_{\alpha n})=R_\alpha$, $e_\alpha(\mathrm{e}^{\mathrm{i}\varphi_{\alpha n}})=e_\alpha$, and $e_\alpha(\mathrm{e}^{-\mathrm{i}\varphi_{\alpha n}})=e^*_\alpha$. For $n=1$, $V_0=e_1(u_1)$ can be absorbed into the single charged lepton field, leaving no free physical parameters in V^{C} as we found earlier. For $n=2$, we have

$$V_0=e_2(u_2)e_1R_1(\theta_1)e^*_1\,, \tag{41}$$

where $e_2(u_2)e_1$ can be absorbed into the two charged lepton fields, while e^*_1 can be pushed through U and absorbed into the massless neutrino field, leaving behind one physical mixing angle in this case. For $n\geq 3$, we combine the two adjacent ω as follows:

$$\begin{aligned}\omega_\alpha\omega_{\alpha+1}&=e_\alpha R_\alpha e^*_\alpha e_{\alpha+1}R_{\alpha+1}e^*_{\alpha+1}\\&=e_\alpha e_{\alpha+1}R_\alpha R_{\alpha+1}e^*_\alpha e^*_{\alpha+1}\,,\end{aligned} \tag{42}$$

because $[e_\alpha,e_\beta]=0$, and $[e_\alpha,R_\beta]=0$ for $\alpha\neq\beta$ and $\alpha<n$. The sequence can be continued such that

$$V_0=\left[e_n(u_n)\prod_{\alpha=1}^{n-1}e_\alpha\right]\cdot\prod_{\beta=1}^{n-1}R_\beta\cdot\left[\prod_{\gamma=1}^{n-1}e^*_\gamma\right]. \tag{43}$$

Again, the right factor commutes with U to get absorbed by the $(n-1)$ massless neutrino fields and the left one by the n charged lepton fields.

To summarize, the lepton mixing matrix V^{C} contains $(n-1)$ free physical mixing angles and no CP-violating phases, and it may be parametrized as in (30) with U given in (32) and

$$V=\prod_{\alpha=1}^{n-1}R_{\alpha n}(\theta_\alpha)\,. \tag{44}$$

The angles θ_α describe the mixing of the $(n-1)$ massless neutrinos with the subsystem of the two massive ones. The mixing matrix V^{N} in NC interactions of neutrinos does not contain these free parameters but is fixed by the two neutrino masses. It is off-diagonal only in the massive subsystem and also conserves CP. It is interesting that the above counting for V^{C} happens to be the same as the one in [26, 27], where the bare Majorana mass terms were not included. In that case, mixing occurs only amongst the n left-handed neutrinos that belong to the leptonic doublets, and there are one massive Dirac neutrino and $(n-1)$ massless neutrinos. Since a massive Dirac neutrino may be considered as a pair of Majorana neutrinos with identical mass, the model studied in [26, 27] appears as a special case in the current work. The Majorana mass terms were indeed included in [25], but it was found that there are $(2n-1)$ mixing angles and n CP-violating phases in V^{C}, totaling $(3n-1)$ free parameters; much more than the number found here. The difference arises from the fact that the constraints (22) from the texture zero in the original neutrino mass matrix have been completely exploited here to remove all unphysical parameters, while they were only partially applied in [25] to delete unitary transformations within the massless neutrinos.

The above counting of physical parameters can be extended to the general case. Without giving further details, we record the results as follows. For $n_0\geq n\geq 1$, V^{C} contains $n(n_0-1)$ mixing angles and $n(n_0-1)$ CP phases, to be compared with $n_0n+n(n-1)/2$ angles and $n_0n+n(n-1)/2$ phases in [25]. Out of them, only $n(2n_0-n-1)/2$ angles and $n(2n_0-n-1)/2$ phases appear in V^{N}. When $n\geq n_0\geq 1$, V^{C} has $n_0(n-1)$ angles and $n(n_0-1)$ phases, much less than the $2n_0n-n_0(n_0+1)/2$ angles and $2n_0n-n-n_0(n_0-1)/2$ phases found in [25]. Out of those, only $n_0(n_0-1)/2$ angles and $n_0(n_0-1)/2$ phases enter in V_{N}.

3 Phenomenological implications

We study in this section some phenomenological implications of the leptonic CC interactions as revealed in (30), (32) and (44). A possible way to measure the absolute neutrino mass is to study the electron spectrum in tritium β decay, which is sensitive to the effective neutrino mass,

$$\begin{aligned}m_{\nu_e}&=\sqrt{\sum_j m_j^2|V^{\mathrm{C}}_{1j}|^2}=|V_{1n}|\sqrt{m_-^2c_m^2+m_+^2s_m^2}\\&=|V_{1n}|\sqrt{m_+m_-}=|V_{1n}||D|\,.\end{aligned} \tag{45}$$

Thus the decay spectrum is only sensitive to a Dirac-type mass in this model.

The neutrinoless double β decay of nuclei is currently the only known practical means to unravel the Majorana nature of neutrinos. At leading order of the expansion in neutrino mass over momentum transfer, the decay amplitude is proportional to the effective neutrino mass,

$$\begin{aligned}m_{\beta\beta}&=\left|\sum_j m_j(V^{\mathrm{C}}_{1j})^2\right|\\&=|V_{1n}|^2|m_-(c_m)^2+m_+(\pm\mathrm{i}s_m)^2|=0\,.\end{aligned} \tag{46}$$

Although the Majorana nature of neutrinos in principle allows the decay to occur, it is highly suppressed in the model considered.

Finally, we want to study whether the model can accommodate the neutrino oscillation data excluding LSND[1]. This would require at least two mass-squared differences and three mixing angles. From our above analysis,

[1] After submitting the paper to the arXiv, I was informed of the earlier work [38], where the neutrino mass matrix was diagonalized explicitly for the case $n=3$, $n_0=1$. The diagonalizing 4×4 unitary matrix was used to compute the neutrino oscillation probability which thus normalizes to unity. However, as emphasized here and in [25], what really appears in the probability for a CC process is the rectangular 3×4 matrix in CC interactions instead of the above diagonalizing matrix even in the basis where the charged leptons are diagonalized. Nevertheless, their phenomenological conclusions for the case $n=3$, $n_0=1$ on the model's incapability of accommodating solar and atmospheric neutrino data are consistent with ours for any n.

we know that with a single sterile neutrino there are only two free mixing angles for three generations. Although this is augmented by an effective mixing angle formed by the two neutrino masses, the chance to accommodate all data looks small. Our analysis below makes this claim stronger. As a first attempt, it is enough to consider the data that may be reasonably well described by vacuum neutrino oscillations, that is, those by KamLAND, K2K and CHOOZ. These experiments produce and detect neutrinos by CC interactions. The amplitude for the whole process that produces a neutrino and a charged lepton $\bar{\ell}_\alpha$ at the source and detects a charged lepton ℓ_β at the detector is proportional to $\sum_j V^{\mathrm{C}*}_{\alpha j} V^{\mathrm{C}}_{\beta j} \exp[-\mathrm{i} m_j^2 L/(2E)]$, where L is the source–detector distance and E the energy of a relativistic neutrino. The probability is

$$P(\alpha \to \beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re}(Q_{\alpha\beta;ij}) \sin^2 \frac{\Delta_{ij} L}{4E} + 2\sum_{i>j} \mathrm{Im}(Q_{\alpha\beta;ij}) \sin \frac{\Delta_{ij} L}{2E}\,, \qquad (47)$$

where $\Delta_{ij} = m_i^2 - m_j^2$ and $Q_{\alpha\beta;ij} = V^{\mathrm{C}*}_{\alpha i} V^{\mathrm{C}}_{\beta i} V^{\mathrm{C}*}_{\beta j} V^{\mathrm{C}}_{\alpha j}$.

The above formula simplifies considerably in our model due to the special structure of V^{C} and the existence of only three different masses. Equations (30) and (32) give

$$V^{\mathrm{C}}_{\alpha j} V^{\mathrm{C}*}_{\beta j} = \sum_{\gamma=1}^{n-1} V_{\alpha\gamma} V^*_{\beta\gamma} \delta_{j\gamma} + V_{\alpha n} V^*_{\beta n} (c_m^2 \delta_{jn} + s_m^2 \delta_{j,n+1})\,, \qquad (48)$$

where the i factor in U drops out already. Taking into account the three different masses, we only need the following quantities:

$$\begin{aligned} \sum_{j=1}^{n-1} Q_{\alpha\beta;n,j} &= c_m^2 |V_{\alpha n}|^2 (\delta_{\alpha\beta} - |V_{\beta n}|^2), \\ \sum_{j=1}^{n-1} Q_{\alpha\beta;n+1,j} &= s_m^2 |V_{\alpha n}|^2 (\delta_{\alpha\beta} - |V_{\beta n}|^2), \\ Q_{\alpha\beta;n+1,n} &= c_m^2 s_m^2 |V_{\alpha n} V_{\beta n}|^2, \end{aligned} \qquad (49)$$

where unitarity of V is used in the first two equalities. They are real even without using the reality of V in (44). Thus the probability for the associated process with $\bar{\ell}_\alpha$, ℓ_β replaced by ℓ_α, $\bar{\ell}_\beta$ is the same. Denoting $x_\pm = m_\pm^2 L/(4E)$, we obtain

$$\begin{aligned} P(\alpha \to \beta) = \delta_{\alpha\beta} &- 4|V_{\alpha n} V_{\beta n}|^2 c_m^2 s_m^2 \sin^2(x_+ - x_-) \\ &- 4|V_{\alpha n}|^2 \left(\delta_{\alpha\beta} - |V_{\beta n}|^2\right) \\ &\times (c_m^2 \sin^2 x_- + s_m^2 \sin^2 x_+)\,. \end{aligned} \qquad (50)$$

Note that the probability does not normalize to unity [25]:

$$\sum_\beta P(\alpha \to \beta) = 1 - 4|V_{\alpha n}|^2 (c_m s_m)^2 \sin^2(x_+ - x_-)\,. \qquad (51)$$

This occurs because the mixing matrix appearing in the amplitude is not the one relating weak and mass eigenstates of neutrinos which is always unitary and would guarantee unity normalization, but the one appearing in CC interactions which is not unitary in the current model, so that the sterile degree of freedom gets effectively lost in the sum.

The KamLAND and CHOOZ are reactor $\bar{\nu}_e$ disappearance experiments at $L/E \sim (20\text{–}50)10^3\,\mathrm{km/GeV}$ and $L/E \sim 333\,\mathrm{km/GeV}$, respectively, and are thus potentially sensitive to small and large mass-squared differences. The K2K experiment observes the accelerator ν_μ disappearance at $L/E \sim 200\,\mathrm{km/GeV}$, close to the range at CHOOZ though in a different channel. That KamLAND and K2K observed deficits and spectral distortions implies that there would be two well-separated mass-squared differences. Then the null result by CHOOZ would be interpreted by some small mixing parameter. There are two ways to arrange this, either (1) $m_+^2 \sim m_-^2 \gg m_+^2 - m_-^2$ or (2) $m_+^2 \gg m_-^2$, implying correspondingly (1) $c_m \sim s_m \sim 1/\sqrt{2}$ or (2) $c_m \sim 1,\ s_m \sim 0$.

In case (2), oscillations in the larger mass-squared difference, i.e., in $x_+ \sim x_+ - x_-$, are highly suppressed in all channels, and thus they are incapable of accommodating the K2K data. In case (1), we have

$$\begin{aligned} P(\alpha \to \alpha) \sim 1 &- 4|V_{\alpha n}|^2 (1 - |V_{\alpha n}|^2) \sin^2 x_+ \\ &- |V_{\alpha n}|^4 \sin^2(x_+ - x_-)\,. \end{aligned} \qquad (52)$$

For K2K and CHOOZ, the last term can be ignored so that their results imply that $4|V_{2n}|^2(1 - |V_{2n}|^2) \sim 1$ and $4|V_{1n}|^2(1 - |V_{1n}|^2) \sim 0$. Since unitarity of V demands that $|V_{1n}|^2 + |V_{2n}|^2 \leq 1$, the combined solution is $|V_{2n}|^2 \sim 1/2$ and $|V_{1n}|^2 \sim 0$. But the latter is rejected by the KamLAND data.

4 Conclusion

We have investigated the leptonic mixing matrices in SM augmented by one sterile neutrino. We found that the mixing matrix V^{C} in CC interactions takes a factorized form, with one factor U describing the mixing in the subsystem of the two massive neutrinos and the other V the mixing of the massless neutrinos with the subsystem. We showed that this arises from the texture zero in the neutrino mass matrix that is protected by gauge symmetry. The matrix U is completely fixed by the two masses, while the factorization makes it possible to remove all phases in V, thus leaving us with $(n-1)$ free physical mixing angles in V^{C} for n generations. The factorization also determines uniquely the mixing matrix V^{N} in NC neutrino interactions in terms of the masses so that off-diagonal interactions occur only between the massive neutrinos. Both CC and NC interactions automatically conserve CP. We also considered some phenomenological results from the exposed structure in V^{C}. The effective neutrino mass in tritium β decay is essentially sensitive to the Dirac mass in the model, while the leading contribution to neutrinoless double β decay vanishes. The difficulty with this simple model is that, even with any number of generations, it cannot accommodate the vacuum neutrino oscillation data coming from KamLAND, K2K and CHOOZ. A way out might be to introduce two sterile neutrinos, which would bring in more free parameters into the model. While the involvement of sterile neutrinos is not favored in solar and atmospheric data, it might still be phenomenologically viable by relaxing the tension between the LSND and other short baseline experiments, if either of the two can be unambiguously confirmed by MiniBooNE.

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